Journal of Approximation Theory **92**, 167–190 (1998)
Article No. AT973125

# Determination of the Jumps of a Bounded Function by Its Fourier Series

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*Communicated by Doron S. Lubinsky*

Received October 26, 1995; accepted in revised form January 28, 1997

The well-known identity which determines the jumps of a function of bounded variation by its Fourier series is extended to larger classes of functions, such as $V_\Phi$, $\Lambda BV$, and $V[v]$, under some conditions on the generalized variations. It is shown as well that the conditions on the generalized variations are definitive in some sense. Based on the above-mentioned results, an identity which determines the jumps of a bounded function by its Fourier series with respect to the system of generalized Jacobi polynomials is obtained for these function classes. © 1998 Academic Press

## INTRODUCTION

**1**. Throughout this paper we use the following general notations: $N$, $Z_+$, and $Z$ are the sets of positive integers, nonnegative integers, and integers, respectively. $L[a, b]$ is the space of integrable functions on $[a, b]$. $W[a, b]$ is the space of functions on $[a, b]$ which may have discontinuities only of the first kind and which are normalized by the condition $f(x) = (f(x+) + f(x-))/2$ (here and elsewhere $f(x+)$ and $f(x-)$ denote the right- and left-hand-side limits of the function $f$ at a point $x$). In addition, we assume that $f \in W[a, b]$ is continuous at the end points of the interval $[a, b]$, i.e., $f(a) = f(a+)$ and $f(b) = f(b-)$, whenever $f$ is not periodically continued outside of the interval. $C[a, b]$ is the space of continuous functions on $[a, b]$ with the uniform norm $\|\cdot\|_{C[a,b]}$ and $H_n$ is the set of algebraic polynomials of degree at most $n$ ($n \in Z_+$) with real coefficients.

$$\omega(f; \delta; [a, b]) = \max\{|f(x) - f(t)| : x, t \in [a, b] \text{ and } |x - t| \leqslant \delta\} \qquad (1)$$

is the modulus of continuity of $f \in C[a, b]$ on $[a, b]$.

If $g \in L[-\pi, \pi]$, $g$ has a Fourier series with respect to the trigonometric system $\{1, \cos n\theta, \sin n\theta\}_{n=1}^{\infty}$, and we denote the $n$th partial sum of the



0021-9045/98 $25.00

Fourier series of $g$ by $S_n(g; \theta)$. By $\tilde{S}_n(g; \theta)$ we denote the $n$th partial sum of the conjugate series, i.e.,

$$\tilde{S}_n(g; \theta) = \sum_{k=1}^{n} (a_k(g) \sin k\theta - b_k(g) \cos k\theta),$$

where $a_k(g)$ and $b_k(g)$ are the Fourier coefficients of the function $g$.

We say that $\mathbf{w}$ is a generalized Jacobi weight, i.e., $\mathbf{w} \in \mathrm{GJ}$, if

$$\mathbf{w}(t) = h(t)(1-t)^{\alpha} (1+t)^{\beta} |t - x_1|^{\delta_1} \cdots |t - x_M|^{\delta_M}, \tag{2}$$

$$h \in C[-1, 1], \quad h(t) > 0 \quad (|t| \leqslant 1), \quad \omega(h; t; [-1, 1]) \, t^{-1} \in L[0, 1], \tag{3}$$

$$-1 < x_1 < \cdots < x_M < 1, \qquad \alpha, \beta, \delta_1, ..., \delta_M > -1. \tag{4}$$

By $\sigma(\mathbf{w}) = (P_n(\mathbf{w}; x))_{n=0}^{\infty}$ we denote the system of algebraic polynomials $P_n(\mathbf{w}; x) = \gamma_n(\mathbf{w}) \, x^n +$ *lower degree terms* with positive leading coefficients $\gamma_n(\mathbf{w})$, which are orthonormal on $[-1, 1]$ with respect to the weight $\mathbf{w} \in \mathrm{GJ}$, i.e.,

$$\int_{-1}^{1} P_n(\mathbf{w}; t) \, P_m(\mathbf{w}; t) \, \mathbf{w}(t) \, dt = \delta_{nm}.$$

Such polynomials are called the generalized Jacobi polynomials and their properties are discussed, for example, in [2, 3, and 11].

If $f\mathbf{w} \in L[-1, 1]$, $\mathbf{w} \in \mathrm{GJ}$, then $f$ has a Fourier series with respect to the system $\sigma(\mathbf{w})$, and by $S_n(\mathbf{w}; f; x)$ we denote the $n$th partial sum of the Fourier series of $f$ with respect to the system $\sigma(\mathbf{w})$, i.e.,

$$S_n(\mathbf{w}; f; x) = \sum_{k=0}^{n-1} a_k(\mathbf{w}; f) \, P_k(\mathbf{w}; x) = \int_{-1}^{1} f(t) \, K_n(\mathbf{w}; x; t) \, \mathbf{w}(t) \, dt,$$

where

$$a_k(\mathbf{w}; f) = \int_{-1}^{1} f(t) \, P_k(\mathbf{w}; t) \, \mathbf{w}(t) \, dt$$

is the $k$th Fourier coefficient of the function $f$, and

$$K_n(\mathbf{w}; x; t) = \sum_{k=0}^{n-1} P_k(\mathbf{w}; x) \, P_k(\mathbf{w}; t)$$

is the Dirichlet kernel of the system $\sigma(\mathbf{w})$.

When $h(t) \equiv 1$, $|t| \leqslant 1$, and $M = 0$ (i.e., a weight does not have singularities strictly inside of the segment $(-1, 1)$), $\mathbf{w} \in \mathrm{GJ}$ is called a Jacobi weight, and in this case we use the commonly accepted notation "$(\alpha, \beta)$" instead of "$\mathbf{w}$"

throughout. For example, we write $S_n^{(\alpha,\beta)}(f;x)$ instead of $S_n(\mathbf{w};f;x)$, and so on.

By $K$ we denote positive constants, possibly depending on some fixed parameters, and in general distinct in different formulae. Sometimes the important arguments of $K$ will be written explicitly in the expressions for it. For positive quantities $A_n$ and $B_n$, possibly depending on some other variables as well, we write $A_n = o(B_n)$ and $A_n = O(B_n)$, if $\lim_{n\to\infty} A_n/B_n = 0$ and $\sup_{n\in N} A_n/B_n < \infty$, respectively.

The following are some generalizations of the notion of bounded variation of a function.

Definition 1 [21]. Let $\Phi$ be a strictly increasing continuous function on $[0,\infty)$ and $\Phi(0)=0$. A function $f$ is said to have $\Phi$-bounded variation on $[a,b]$, i.e., $f\in V_\Phi[a,b]$, if

$$v_\Phi(f;[a,b]) = \sup_{\Pi} \sum_{k=1}^{n} \Phi(|f(x_k)-f(x_{k-1})|) < \infty,$$

where $\Pi = \{a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b\}$ is an arbitrary partition of $[a,b]$.

If $\Phi(x)=x$, then $V_\Phi[a,b]$ coincides with the Jordan class $V[a,b]$ of functions of bounded variation, and if $\Phi(x)=x^p$, $p>1$, it coincides with the Wiener [20] class $V_p[a,b]$.

Definition 2 [18]. Let $\Lambda = (\lambda_k)_{k=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$. A function $f$ is said to have $\Lambda$-bounded variation on $[a,b]$, i.e., $f\in \Lambda BV[a,b]$, if

$$v_\Lambda(f;[a,b]) = \sup_{\Pi} \sum_{k=0}^{n} \frac{|f(x_{2k+1})-f(x_{2k})|}{\lambda_k} < \infty,$$

where $\Pi$ is an arbitrary system of disjoint intervals $(x_{2k}, x_{2k+1}) \subset [a,b]$.

If $\lambda_k = 1$, $k\in N$, then obviously $\Lambda BV[a,b] = V[a,b]$.

We say that a function $f$ is of *harmonic bounded variation* on $[a,b]$, i.e., $f\in HBV[a,b]$, if $\lambda_k = k$, $k\in N$.

Definition 3 [5]. Let $f$ be a bounded function defined on $[a,b]$. The modulus of variation of the function $f$ is the function $v(n;f;[a,b])$ defined for $n\in Z_+$ as follows: $v(0;f;[a,b])=0$, while for $n\geqslant 1$,

$$v(n;f;[a,b]) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(x_{2k+1})-f(x_{2k})|,$$

where $\Pi_n$ is an arbitrary system of $n$ disjoint intervals $(x_{2k}, x_{2k+1}) \subset [a, b]$, $k = 0, 1, ..., n-1$.

If $v(n)$, $n \in N$, is a nondecreasing upwards convex function and $v(0) = 0$, then we call $v(n)$ a modulus of variation.

The class of functions which satisfy the relation $v(n; f; [a, b]) = O(v(n))$ will be denoted by $V[v; [a, b]]$.

In particular, $V[1; [a, b]] = V[a, b]$.

DEFINITION 4 (cf. [22, p. 16]). We say that a function $\Phi$ has the complementary function $\Psi$ in the sense of W. H. Young, if

$$\Phi(x) = \int_0^x \phi(t)\, dt \qquad \text{and} \qquad \Psi(x) = \int_0^x \psi(t)\, dt,$$

where $\phi$ is a strictly increasing continuous function on $[0, \infty)$, $\phi(0) = 0$, and $\psi(x) = \phi^{-1}(x)$ for $x \in [0, \infty)$.

If there is no ambiguity, we shall usually omit the dependence on the domain and simply refer to one of the introduced classes of functions or the quantities as $L$, $W$, ..., or $v_\Phi(f)$, $v_\Lambda(f)$, etc.

**2.** The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time:

THEOREM FC ([4, 7]). *Let $g \in V$ be a $2\pi$-periodic function. Then the following identity*

$$\lim_{n \to \infty} \frac{(S_n(g; \theta))'}{n} = \frac{1}{\pi} (g(\theta+) - g(\theta-)) \tag{5}$$

*is valid for every fixed $\theta \in [-\pi, \pi]$.*

Let us mention that the jumps of $g \in L$ can be determined directly from its conjugate Fourier partial sums as well (cf. [22, Theorem (8.13), p. 60]).

Golubov [9] generalized identity (5) for $V_p$ classes of functions and higher derivatives of the Fourier partial sums.

THEOREM G1 [9, Theorem 1, p. 444]. *Let $r \in Z_+$ and suppose $g \in V_p$ for some $p \geqslant 1$. Then*

(a) *the identity*

$$\lim_{n \to \infty} \frac{(S_n(g; \theta))^{(2r+1)}}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\,\pi} (g(\theta+) - g(\theta-)) \tag{6}$$

*is valid for every fixed $\theta \in [-\pi, \pi]$.*

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in V_p$, $p \geqslant 1$, *by means of the sequence* $((S_n(g;\theta))^{(2r)})_{n=0}^{\infty}$.

THEOREM G2 [9, Theorem 2, p. 445]. *Let* $r \in N$ *and suppose* $g \in V_p$ *for some* $p \geqslant 1$. *Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{(\tilde{S}_n(g;\theta))^{(2r)}}{n^{2r}} = \frac{(-1)^{r+1}}{2r\pi}(g(\theta+)-g(\theta-)) \tag{7}$$

*is valid for every fixed* $\theta \in [-\pi, \pi]$.

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in V_p$, $p \geqslant 1$, *by means of the sequence* $((\tilde{S}_n(g;\theta))^{(2r-1)})_{n=1}^{\infty}$.

## MAIN RESULTS

In the present paper we extend identities (6) and (7) to the larger classes of functions such as $V_\Phi$, $\Lambda BV$, and $V[v]$, under some conditions on the generalized variations of a function. We investigate definitiveness of these conditions as well.

Furthermore, we establish an identity which determines the jumps of a bounded function by means of its differentiated Fourier partial sums with respect to the system of generalized Jacobi polynomials. The identity is studied for the same classes of functions of generalized bounded variation.

THEOREM 1. *Let* $r \in Z_+$ *and suppose* $\Lambda BV$ *is the class of functions of* $\Lambda$*-bounded variation determined by the sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$. *Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{(S_n(g;\theta))^{(2r+1)}}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(g(\theta+)-g(\theta-)) \tag{8}$$

*is valid for every* $g \in \Lambda BV$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if*

$$\Lambda BV \subseteq HBV. \tag{9}$$

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in \Lambda BV$ *by means of the sequence* $((S_n(g;\theta))^{(2r)})_{n=0}^{\infty}$.

THEOREM 2. *Let* $r \in Z_+$ *and suppose* $V[v]$ *is the class of functions determined by a given modulus of variation* $v(n)$. *Then*

(a) *identity* (8) *is valid for every* $g \in V[v]$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if*

$$\sum_{k=1}^{\infty} \frac{v(k)}{k^2} < \infty. \tag{10}$$

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in V[v]$ *by means of the sequence* $((S_n(g; \theta))^{(2r)})_{n=0}^{\infty}$.

THEOREM 3. *Let* $r \in Z_+$ *and suppose* $V_\Phi$ *is the class of functions of* $\Phi$-*bounded variation, where* $\Phi$ *has the complementary function* $\Psi$ *in the sense of Young* (*see Definition* 4). *Then*

(a) *identity* (8) *is valid for every* $g \in V_\Phi$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if*

$$\sum_{k=1}^{\infty} \Psi(1/k) < \infty, \tag{11}$$

$$\int_0^1 \ln(1/\Phi(x))\, dx < \infty, \tag{12}$$

*or*

$$\sum_{k=1}^{\infty} \frac{1}{k} \Phi^{-1}(1/k) < \infty, \tag{13}$$

*where* $\Phi^{-1}$ *is inverse of* $\Phi$.

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in V_\Phi$ *by means of the sequence* $((S_n(g; \theta))^{(2r)})_{n=0}^{\infty}$.

The following three theorems are generalizations of Theorem G2.

THEOREM 4. *Let* $r \in N$ *and suppose* $\Lambda BV$ *is the class of functions of* $\Lambda$-*bounded variation determined by the sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$. *Then*

(a) *the identity*

$$\lim_{n \to \infty} \frac{(\tilde{S}_n(g; \theta))^{(2r)}}{n^{2r}} = \frac{(-1)^{(r+1)}}{2r\pi} (g(\theta+) - g(\theta-)) \tag{14}$$

*is valid for every* $g \in \Lambda BV$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if condition* (9) *holds.*

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in \Lambda BV$ *by means of the sequence* $((\tilde{S}_n(g; \theta))^{(2r-1)})_{n=1}^{\infty}$.

THEOREM 5. *Let* $r \in N$ *and suppose* $V[v]$ *is the class of functions determined by a given modulus of variation* $v(n)$. *Then*

(a) *the identity* (14) *is valid for every* $g \in V[v]$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if condition* (10) *holds.*

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in V[v]$ *by means of the sequence* $((\tilde{S}_n(g; \theta))^{(2r-1)})_{n=1}^{\infty}$.

THEOREM 6. *Let* $r \in N$ *and suppose* $V_\Phi$ *is the class of functions of* $\Phi$*-bounded variation, where* $\Phi$ *has the complementary function* $\Psi$ *in the sense of Young. Then*

(a) *identity* (14) *is valid for every* $g \in V_\Phi$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if condition* (11), (12), *or* (13) *holds.*

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in V_\Phi$ *by means of the sequence* $((\tilde{S}_n(g; \theta))^{(2r-1)})_{n=1}^{\infty}$.

The following theorems establish an identity which determines the jumps of a bounded function by means of its differentiated Fourier partial sums with respect to the system of generalized Jacobi polynomials.

THEOREM 7. *Let* $r \in Z_+$, $\mathbf{w} \in \mathrm{GJ}$, *and suppose* $\Lambda BV$ *is the class of functions of* $\Lambda$*-bounded variation determined by the sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$. *Then the identity*

$$\lim_{n \to \infty} \frac{(S_n(\mathbf{w}; f; x))^{(2r+1)}}{n^{2r+1}} = \frac{(-1)^r (1-x^2)^{-r-1/2}}{(2r+1)\pi} (f(x+) - f(x-)) \tag{15}$$

*is valid for every* $f \in \Lambda BV$ *and each fixed* $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, *if condition* (9) *holds.*

*If, in addition, the weight* $\mathbf{w} \in \mathrm{GJ}$ *satisfies the following conditions*:

$$\alpha \geqslant -1/2, \quad \beta \geqslant -1/2, \quad \delta_1 \geqslant 0, ..., \delta_M \geqslant 0, \quad \omega(h; t)\, t^{-1} \ln t \in L[0, 1], \tag{16}$$

*then condition* (9) *is necessary for the validity of identity* (15) *for every* $f \in \Lambda BV$ *and each fixed* $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, *as well.*

THEOREM 8. *Let* $r \in Z_+$, $\mathbf{w} \in \mathrm{GJ}$, *and suppose* $V[v]$ *is the class of functions determined by a modulus of variation* $v(n)$. *Then identity* (15) *is* *valid for every* $f \in V[v]$ *and each fixed* $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, *if condition* (10) *holds.*

*If, in addition, the weight* $\mathbf{w} \in \mathrm{GJ}$ *satisfies condition* (16), *then condition* (10) *is necessary for the validity of identity* (15) *for every* $f \in V[v]$ *and each fixed* $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, *as well.*

THEOREM 9. *Let* $r \in Z_+$, $\mathbf{w} \in \mathrm{GJ}$, *and suppose* $V_\Phi$ *is the class of functions of* $\Phi$*-bounded variation, where* $\Phi$ *has the complementary function* $\Psi$ *in the sense of Young. Then identity* (15) *is valid for every* $f \in V_\Phi$ *and each fixed* $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, *if condition* (11), (12), *or* (13) *holds.*

*If, in addition, the weight* $\mathbf{w} \in \mathrm{GJ}$ *satisfies condition* (16), *then conditions* (11), (12), *and* (13) *are necessary for the validity of identity* (15) *for every* $f \in V_\Phi$ *and each fixed* $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, *as well.*

Theorems 7, 8, and 9 imply criteria for the continuity of a function of $V_\Phi$, $\Lambda BV$, and $V[v]$ classes analogous to the Wiener's criterion for the continuity of a function of bounded variation (see [20, p. 81] or [22, Thm. (9.6), p. 108]).

COROLLARY 1. *Let* $\mathbf{w} \in \mathrm{GJ}$ *and suppose* $f \in HBV$. *If the Fourier coefficients of the function* $f$ *with respect to the system* $\sigma(\mathbf{w})$ *for some* $r \in Z_+$ *satisfy the condition*

$$\sum_{k=1}^{n} k^{2r+1} \, |a_k(\mathbf{w}; f)| = o(n^{2r+1}), \tag{17}$$

*then* $f$ *is continuous at each point* $x \in [-1, 1]$, $x \neq x_1, ..., x_M$.

*In particular, if a weight* $\mathbf{w} \in \mathrm{GJ}$ *is such that*

$$M = 0, \tag{18}$$

*then condition* (17) *implies that* $f \in C$.

COROLLARY 2. *Let* $\mathbf{w} \in \mathrm{GJ}$ *and suppose* $f \in V[v]$, *where* $v(n)$ *satisfies condition* (10). *If the Fourier coefficients of* $f$ *with respect to* $\sigma(\mathbf{w})$ *for some* $r \in Z_+$ *satisfy condition* (17), *then* $f$ *is continuous at each point* $x \in [-1, 1]$, $x \neq x_1, ..., x_M$.

*In particular, if a weight* $\mathbf{w} \in \mathrm{GJ}$ *satisfies condition* (18), *then condition* (17) *implies that* $f \in C$.

COROLLARY 3. *Let* $\mathbf{w} \in \mathrm{GJ}$ *and suppose* $f \in V_\Phi$, *where* $\Phi$ *satisfies condition* (11), (12), *or* (13). *If the Fourier coefficients of* $f$ *with respect to* $\sigma(\mathbf{w})$ *for some* $r \in Z_+$ *satisfy condition* (17), *then* $f$ *is continuous at each point* $x \in [-1, 1]$, $x \neq x_1, ..., x_M$.

*In particular, if a weight* $\mathbf{w} \in \mathrm{GJ}$ *satisfies condition* (18), *then condition* (17) *implies that* $f \in C$.

## PRELIMINARIES

For a given weight $\mathbf{w} \in \mathrm{GJ}$ we always assume that $x_0 = -1$ and $x_{M+1} = 1$. In addition, for a fixed $\varepsilon \in (0, (x_{\nu+1} - x_\nu)/2)$, $\nu = 0, 1, ..., M$, we set

$$\Delta(\nu; \varepsilon) = [x_\nu + \varepsilon; x_{\nu+1} - \varepsilon]. \tag{19}$$

If $K_n(\mathbf{w}; x; t)$ is the Dirichlet kernel of the system $\sigma(\mathbf{w})$, then the Christoffel–Darboux formula

$$\begin{aligned}(x-t)\, K_n(\mathbf{w}; x; t) = (\gamma_{n-1}(\mathbf{w})/\gamma_n(\mathbf{w}))(&P_n(\mathbf{w}; x)\, P_{n-1}(\mathbf{w}; t) \\ &- P_{n-1}(\mathbf{w}; x)\, P_n(\mathbf{w}; t))\end{aligned} \tag{20}$$

holds for $x, t \in [-1, 1]$ and $n \in N$, where $\gamma_{n-1}(\mathbf{w})/\gamma_n(\mathbf{w}) \leqslant 2$ for $n \in N$ (cf. [8, formula (4.3), p. 24 and Lemma 7.2, p. 41]).

Let $\mathbf{w} \in \mathrm{GJ}$. Then the inequality

$$\begin{aligned}|P_{n-1}(\mathbf{w}; x)| < K(\mathbf{w}) &\left(\sqrt{1-x} + \frac{1}{n}\right)^{-\alpha - 1/2} \left(\sqrt{1+x} + \frac{1}{n}\right)^{-\beta - 1/2} \\ &\times \prod_{\nu=1}^{M} \left(|x - x_\nu| + \frac{1}{n}\right)^{-\delta_\nu/2}\end{aligned} \tag{21}$$

holds for $x \in [-1, 1]$ and $n \in N$ [2, Coro. 2.1, p. 25].

Let the weight $\rho \in \mathrm{GJ}$ be such that $h(t) \equiv 1$, $|t| \leqslant 1$. Then the asymptotic formula [3, Coro. 12, p. 38]

$$P_n(\rho; \cos\theta) = (2/[\pi\varphi(\theta)])^{1/2}\, (\cos[n\theta - \gamma(\theta)] + o(1)) \tag{22}$$

holds as $n \to \infty$, uniformly for $\cos\theta \in \Delta(\nu; \varepsilon)$, $\varepsilon \in (0, (x_{\nu+1} - x_\nu)/2)$, and $\nu = 0, 1, ..., M$, where $\gamma(\theta) = (\alpha + \beta + \delta_1 + \cdots + \delta_M + 1)\,\theta/2 - (\alpha/2 + 1/4)\,\pi$ and $\varphi(\theta) = \rho(\cos\theta)\,|\sin\theta|$.

PROPOSITION. *Let the weight* $\rho \in \mathrm{GJ}$ *be such that* $h(t) \equiv 1$, $|t| \leqslant 1$, *and suppose* $f \in HBV$. *Then for a fixed* $\nu = 0, 1, ..., M$ *and a fixed* $\varepsilon \in (0, (x_{\nu+1} - x_\nu)/2)$ *there exists a constant* $K(f; \nu; \varepsilon)$ *such that*

$$\sup_{n \in N} \|S_n(\rho; f; \cdot)\|_{C[\Delta(\nu; \varepsilon)]} < K(f; \nu; \varepsilon). \tag{23}$$

*Proof.* We have

$$|S_n(\rho; f; x)| \leqslant |S_n(\rho; f; x) - S_n^{(-1/2, -1/2)}(f; x)| \\ + |S_n^{(-1/2, -1/2)}(f; x) - f(x)| + |f(x)| = J_1 + J_2 + J_3. \qquad (24)$$

Taking into account that $f$ is bounded, the estimate

$$J_3 < K(f) \qquad (25)$$

is obvious.

There are numerous results about estimates of the differences $S_n(\mathbf{w}; f; x) - S_n^{(-1/2, -1/2)}(f; x)$ for $\mathbf{w} \in \mathrm{GJ}$ (see [2, Section 4, p. 42] and the indicated references).

The estimate

$$J_1 = \|S_n(\rho; f; \cdot) - S_n^{(-1/2, -1/2)}(f; \cdot)\|_{C[\Delta(v; \varepsilon)]} = o(1) \qquad (26)$$

can be easily deduced from a slight modification of the proof of Theorem 9.1.2 [17, p. 246], utilizing the asymptotic formula (22).

Further, we use the following estimate for a bounded integrable function $f$ [2, Lemma 3.1, p. 33]: Let $x \in [-1, 1]$ and $y = \arccos x$. Then

$$|S_n^{(-1/2, -1/2)}(f; x) - f(x)| \leqslant \frac{n}{\pi^2} \sum_{|k|=1}^{n} \frac{|A_k|}{k} + K \sup_{x \in [-1, 1]} |f(x)| \qquad (27)$$

for $n \in N$, uniformly with respect to $f$ and $x$, where

$$A_k = \int_0^{\pi/(2n+1)} (g(y_k + \tau) - g(y_k - \tau)) \sin \frac{2n+1}{2} \tau \, d\tau, \qquad (28)$$

$$y_k = y + 2\pi k/(2n+1) \qquad (k \in Z), \qquad (29)$$

and the function $g$ is defined by

$$g(\theta) = f(\cos \theta). \qquad (30)$$

Hence by (27) and (28)

$$J_2 \leqslant \frac{n}{\pi^2} \sum_{|k|=1}^{n} \frac{1}{k} \int_0^{\pi/(2n+1)} |g(y_k + \tau) - g(y_k - \tau)| \, d\tau + K \sup_{x \in [-1, 1]} |f(x)| \\ = J_{21} + J_{22}. \qquad (31)$$

Let

$$g_k = \sup_{\tau \in [0, \pi/(2n+1)]} |g(y_k + \tau) - g(y_k - \tau)|. \tag{32}$$

Since for $\tau \in [0, \pi/(2n+1)]$ and $k, m = 1, ..., n$, $k \neq m$, we have

$$(y \pm 2k\pi/(2n+1) - \tau,\ y \pm 2k\pi/(2n+1) + \tau)$$
$$\cap (y \pm 2m\pi/(2n+1) - \tau,\ y \pm 2m\pi/(2n+1) + \tau) = \varnothing,$$

it follows from (31), (32), and Definition 2 that

$$J_{21} \leqslant \frac{n}{(2n+1)\pi} \sum_{|k|=1}^{n} \frac{g_k}{k} \leqslant K v_H(g) = K v_H(f) < \infty, \tag{33}$$

since the variation of a function does not change under a monotonic transformation of a variable.

Consequently, in view of (24–26), (31), and (33) we obtain the desired inequality, and the proposition is proved. ∎

Lemma 1 [16, Theorem 2.5, pp. 429, 430]. *The sets* $C[a, b] \cap V_\Phi[a, b]$, $C[a, b] \cap \Lambda BV[a, b]$, *and* $C[a, b] \cap V[v; [a, b]]$ *form Banach spaces with norms*

$$\|f\|_\Phi = \inf\{\eta > 0 : v_\Phi(f/\eta; [a, b]) \leqslant 1\} + |f(a)|, \tag{34}$$

$$\|f\|_\Lambda = v_\Lambda(f; [a, b]) + |f(a)|, \tag{35}$$

*and*

$$\|f\|_V = \sup_{n \in N} \frac{v(n; f; [a, b])}{v(n)} + |f(a)|, \tag{36}$$

*respectively.*

Regarding (34) and (35) see also [10, p. 32] and [18, p. 108].

Definition 5 [19]. Let $\Lambda(e) = (\lambda_{k+e})_{k=1}^{\infty}$, $e \in N$, where the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$ satisfies the conditions of Definition 2. A function $f \in \Lambda BV$ is said to be continuous in $\Lambda$-variation, i.e, $f \in \Lambda_C BV$, if $v_{\Lambda(e)}(f) = o(1)$ as $e \to \infty$.

LEMMA 2 [15, Theorem 1, p. 88]. *Let a sequence* $\Lambda=(\lambda_k)_{k=1}^{\infty}$, *satisfying the conditions of Definition* 2, *be such that* $\lim_{k\to\infty}\lambda_k/\lambda_{2k}$ *exists. Then* $\Lambda_C BV=\Lambda BV$ *if and only if*

$$\lim_{k\to\infty}\frac{\lambda_k}{\lambda_{2k}}<1.$$

## PROOFS

*Proof of Theorem* 1(a): *Sufficiency.* Let us assume that $\theta\in[-\pi,\pi]$ is fixed and $g\in HBV$. It is known [18, p. 108] that

$$\Lambda BV[a,b]\subset W[a,b]$$

for an arbitrary $\Lambda BV[a,b]$ class. So, the Fourier series of a function $g\in HBV$ is defined.

Obviously, by means of a change of variables, the problem can always be reduced to the case $\theta=0$.

As shown in [9, p. 447] for the saw tooth function $G_0(\theta)=(\pi-\theta)/2\in V$, $0<\theta<2\pi$, and $G_0(\theta)=G_0(\theta+2\pi)$, which has the jump of $\pi$ at the point $\theta=0$, the following identity holds:

$$\lim_{n\to\infty}\frac{(S_n(G_0;0))^{(2r+1)}}{n^{2r+1}}=\frac{(-1)^r}{2r+1}.$$

Next, for a given function $g\in HBV$ with a jump at $\theta=0$, let us set

$$G(\theta)=g(\theta)-\frac{g(\theta+)-g(\theta-)}{\pi}\,G_0(\theta).$$

Obviously $G$ is continuous at $\theta=0$ and $G\in HBV$. We can assume as well that $G(0)=0$ since identity (8) is invariant with respect to subtraction of a constant from a function. Hence, it is left to show that if $g\in HBV$, $g(0)=0$, and $g$ is continuous at $\theta=0$, then

$$\lim_{n\to\infty}\frac{(S_n(g;0))^{(2r+1)}}{n^{2r+1}}=0. \tag{37}$$

If $D_n(\tau)$ is the Dirichlet kernel, i.e.,

$$D_n(\tau)=\frac{1}{2}+\sum_{k=1}^{n}\cos k\tau,$$

then the following representation is valid [22, formula (5.2), p. 50]

$$D_n(\tau)=\frac{\sin n\tau}{\tau}+\xi(\tau)\sin n\tau+\frac{1}{2}\cos n\tau \qquad (|\tau|\leqslant\pi), \tag{38}$$

where $\xi(\tau)=\frac{1}{2}\cot(\tau/2)-(1/\tau)$ for $0<|\tau|\leqslant\pi$ and $\xi(0)=0$. It is known that $\xi$ is analytic on the interval $(-2\pi, 2\pi)$, so $\max_{|\tau|\leqslant\pi}|(\xi(\tau))^{(k)}|<\infty$, $k\in Z_+$.
From (38) follows $(r\in N)$

$$\begin{aligned}(D_n(\tau))^{(r)}=n^r\,&\frac{\sin(n\tau+r\pi/2)}{\tau}\\ &+\sum_{i=0}^{r-1}\binom{r}{i}n^i\sin(n\tau+i\pi/2)(-1)^{r-i}(r-i)!\,\tau^{i-r-1}\\ &+\sum_{i=0}^{r}\binom{r}{i}(\xi(\tau))^{(r-i)}n^i\sin(n\tau+i\pi/2)\\ &+\frac{1}{2}n^r\cos(n\tau+r\pi/2).\end{aligned} \tag{39}$$

Furthermore,

$$\begin{aligned}(S_n(g;0))^{(2r+1)}&=-\frac{1}{\pi}\int_{-\pi}^{\pi}g(\tau)(D_n(\tau))^{(2r+1)}\,d\tau\\ &=-\frac{1}{\pi}\left(\int_{-\pi/2n}^{\pi/2n}+\int_{\pi/2n}^{\pi}+\int_{-\pi}^{-\pi/2n}\right)g(\tau)(D_n(\tau))^{(2r+1)}\,d\tau\\ &=I_1+I_2+I_3.\end{aligned} \tag{40}$$

Let $\varepsilon>0$ be an arbitrary fixed number. Then by virtue of Definition 5 and Lemma 2 there exists $0<\varepsilon^*<1$ such that

$$-\varepsilon^*\ln\varepsilon^*+v_{H([1/\varepsilon^*])}(g)<\varepsilon, \tag{41}$$

where $[a]$ means the integer part of a number $a$. Moreover, since $g$ is continuous at $\theta=0$ and $g(0)=0$, there exists $0<\delta<1$ such that

$$|g(\tau)|<\varepsilon^* \tag{42}$$

whenever $|\tau| < \delta$. Now, let us take $N(\varepsilon) \in N$ so large that

$$\frac{2}{\varepsilon^* n} \sup_{|\tau| \leqslant \pi} (|g(\tau)| + \pi) < \delta \tag{43}$$

holds for $n > N(\varepsilon)$.

Consequently, if $n > N(\varepsilon)$, by (43) we have $\pi/2n < \delta$. So then by (42)

$$\begin{aligned} \frac{|I_1|}{n^{2r+1}} &= \frac{1}{\pi n^{2r+1}} \left| \int_{-\pi/2n}^{\pi/2n} g(\tau)(D_n(\tau))^{(2r+1)}\, d\tau \right| \\ &\leqslant \frac{1}{\pi n^{2r+1}} \int_{-\pi/2n}^{\pi/2n} |g(\tau)|\, |(D_n(\tau))^{(2r+1)}|\, d\tau \\ &\leqslant \frac{1}{\pi n^{2r+1}}\, \varepsilon^* \frac{\pi}{n} \max_{|\tau| \leqslant \pi} |(D_n(\tau))^{(2r+1)}| \\ &\leqslant \frac{\varepsilon^*}{n^{2r+2}} \sum_{k=1}^{n} k^{2r+1} < \varepsilon^*. \end{aligned} \tag{44}$$

In addition, by (39) and (40), we have

$$\begin{aligned} -\frac{\pi I_2}{n^{2r+1}} &= \frac{1}{n^{2r+1}} \int_{\pi/2n}^{\pi} g(\tau)(D_n(\tau))^{(2r+1)}\, d\tau \\ &= (-1)^r \int_{\pi/2n}^{\pi} g(\tau) \frac{\cos n\tau}{\tau}\, d\tau + \sum_{i=0}^{2r} \binom{2r+1}{i} n^{i-2r-1} \\ &\quad \times (-1)^{2r+1-i} (2r+1-i)! \int_{\pi/2n}^{\pi} g(\tau) \frac{\sin(n\tau + i\pi/2)}{\tau^{2r+2-i}}\, d\tau \\ &\quad + \sum_{i=0}^{2r+1} \binom{2r+1}{i} n^{i-2r-1} \int_{\pi/2n}^{\pi} g(\tau)(\xi(\tau))^{(2r+1-i)} \sin(n\tau + i\pi/2)\, d\tau \\ &\quad + \frac{(-1)^{r+1}}{2} \int_{\pi/2n}^{\pi} g(\tau) \sin n\tau\, d\tau \\ &= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned} \tag{45}$$

Let us estimate $I_{21}$. If

$$\theta_{k,n} = \frac{\pi}{2n} + \frac{\pi(k-1)}{n} \tag{46}$$

for $k=1, 2, ..., n$, $\theta_{0,n}=0$, and $\theta_{n+1,n}=\pi$, then by Abel's transformation we get

$$
\begin{aligned}
(-1)^r I_{21} &= \int_{\pi/2n}^{\pi} g(\tau)\,\frac{\cos n\tau}{\tau}\,d\tau \\
&= \sum_{k=1}^{n} \int_{\theta_{k,n}}^{\theta_{k+1,n}} g(\tau)\,\frac{\cos n\tau}{\tau}\,d\tau \\
&= \sum_{k=1}^{n} \int_{\theta_{k,n}}^{\theta_{k+1,n}} (g(\tau)-g(\theta_{k,n}))\,\frac{\cos n\tau}{\tau}\,d\tau \\
&\quad + \sum_{k=1}^{n} g(\theta_{k,n}) \int_{\theta_{k,n}}^{\theta_{k+1,n}} \frac{\cos n\tau}{\tau}\,d\tau \\
&= \sum_{k=1}^{n} \int_{\theta_{k,n}}^{\theta_{k+1,n}} (g(\tau)-g(\theta_{k,n}))\,\frac{\cos n\tau}{\tau}\,d\tau \\
&\quad + \sum_{k=0}^{n-1} (g(\theta_{k+1,n})-g(\theta_{k,n})) \int_{\theta_{k+1,n}}^{\pi} \frac{\cos n\tau}{\tau}\,d\tau. \qquad (47)
\end{aligned}
$$

Since

$$
\left| \int_{\theta_{k,n}}^{\pi} \frac{\cos n\tau}{\tau}\,d\tau \right| \leqslant \left| \int_{\theta_{k,n}}^{\theta_{k+1,n}} \frac{\cos n\tau}{\tau}\,d\tau \right|
$$

and

$$
\int_{\theta_{k,n}}^{\theta_{k+1,n}} \left| \frac{\cos n\tau}{\tau} \right| d\tau \leqslant \frac{2}{k}
$$

for $k=1, 2, ..., n$, then by (42), (43), (46), and (47) we have

$$
\frac{1}{2}\,|I_{21}| \leqslant \sum_{k=1}^{n} \frac{g_{k,n}}{k} + \sum_{k=0}^{n-1} \frac{|g(\theta_{k+1,n})-g(\theta_{k,n})|}{k+1} \leqslant \varepsilon^* + 2\sum_{k=1}^{n} \frac{g_{k,n}}{k}
$$

for $n>N(\varepsilon)$, where $g_{k,n}=\sup_{\tau\in[\theta_{k,n},\,\theta_{k+1,n}]} |g(\tau)-g(\theta_{k,n})|$.

According to Definition 5 we have

$$
\frac{1}{4}\,|I_{21}| < \varepsilon^* + \sum_{k=1}^{m} \frac{g_{k,n}}{k} + \sum_{k=1}^{n-m} \frac{g_{k+m,n}}{k+m} \leqslant \varepsilon^* + \sum_{k=1}^{m} \frac{g_{k,n}}{k} + v_{H(m)}(g)
$$

for an arbitrary $m=1, 2, ..., n-1$.

Let us set $m=[1/\varepsilon^*]$. Then $\theta_{k,n}=\pi/2n+\pi(k-1)/n\leqslant 2\pi[1/\varepsilon^*]/n<\delta$ for $n>N(\varepsilon)$ and $k=1, 2, ..., m+1$ (see (43)). So by (42) $g_{k,n}\leqslant 2\varepsilon^*$, $k=1, 2, ..., m+1$, and taking into account (41) we obtain

$$|I_{21}|\leqslant K\left(\varepsilon^*+\varepsilon^*\sum_{k=1}^{[1/\varepsilon^*]}\frac{1}{k}+v_{H([1/\varepsilon^*])}(g)\right)<K\varepsilon. \tag{48}$$

Now from (45) we have

$$|I_{22}|\leqslant\sum_{i=0}^{2r}\binom{2r+1}{i}n^{i-2r-1}(2r+1-i)!\int_{\pi/2n}^{\pi}\frac{|g(\tau)|}{\tau^{2r+2-i}}\,d\tau.$$

So for $n>N(\varepsilon)$ and $i=0, 1, ..., 2r$, by (42) and (43) we have

$$\begin{aligned}n^{i-2r-1}\int_{\pi/2n}^{\pi}\frac{|g(\tau)|}{\tau^{2r+2-i}}\,d\tau&=n^{i-2r-1}\left(\int_{\pi/2n}^{\delta}+\int_{\delta}^{\pi}\right)\frac{|g(\tau)|}{\tau^{2r+2-i}}\,d\tau\\&\leqslant Kn^{i-2r-1}\left(\varepsilon^* n^{2r+1-i}+\frac{1}{\delta^{2r+1-i}}\sup_{|\tau|\leqslant\pi}|g(\tau)|\right)\\&<K\left(\varepsilon^*+\frac{1}{n\delta}\sup_{|\tau|\leqslant\pi}|g(\tau)|\right)<K\varepsilon^*.\end{aligned}$$

Hence

$$|I_{22}|<K\varepsilon^*\sum_{i=0}^{2r}\binom{2r+1}{i}(2r+1-i)!<K(r)\,\varepsilon^*. \tag{49}$$

It is obvious that by increasing $N(\varepsilon)$, if necessary, in view of (45) and the Riemann–Lebesgue Theorem [14, Thm. (4.4), p. 45], we get

$$\begin{aligned}|I_{23}|+|I_{24}|<&\sum_{i=0}^{2r}\binom{2r+1}{i}n^{i-2r-1}\int_{\pi/2n}^{\pi}|g(\tau)(\xi(\tau))^{(2r+1-i)}|\,d\tau\\&+\left|\int_{\pi/2n}^{\pi}g(\tau)\,\xi(\tau)\cos n\tau\,d\tau\right|\\&+\frac{1}{2}\left|\int_{\pi/2n}^{\pi}g(\tau)\sin n\tau\,d\tau\right|<\varepsilon\end{aligned} \tag{50}$$

for $n>N(\varepsilon)$.

Hence, by virtue of (45), (48), (49), and (50) we have $|I_2|/n^{2r+1} < K(r)\,\varepsilon$. Then by symmetry we get $|I_3|/n^{2r+1} < K(r)\,\varepsilon$ as well. So, taking into account (40) and (44), we obtain

$$\frac{|(S_n(g;0))^{(2r+1)}|}{n^{2r+1}} < K(r)\,\varepsilon$$

for $n > N(\varepsilon)$. Since $\varepsilon$ is arbitrary, (37) follows and sufficiency of assertion (a) of Theorem 1 is proved.

*Necessity.* If condition (9) does not hold, then (see [13, Proof of Thm. 3, p. 116]) there exists a decreasing sequence of positive numbers $(c_k)_{k=1}^{\infty}$, $c_k \to 0$ as $k \to \infty$, such that

$$\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} < \infty \tag{51}$$

and

$$\sum_{k=1}^{\infty} \frac{c_k}{k} = \infty. \tag{52}$$

Let us, for a fixed $r \in Z_+$, consider a sequence of linear functionals $S_n(g) = (S_n(g;0))^{(2r+1)}/n^{2r+1}$, $n \in N$, defined on the Banach space $C \cap \Lambda BV$ with norm (35). We shall show that the sequence of norms of functionals $(\|S_n\|)_{n=1}^{\infty}$ is not bounded. Then the existence of $g_0 \in C \cap \Lambda BV$ such that

$$\limsup_{n \to \infty} \frac{|(S_n(g_0;0))^{(2r+1)}|}{n^{2r+1}} > 0 \tag{53}$$

immediately follows from Banach–Steinhaus Theorem. For this purpose let us define functions $g_n$ as follows:

$$g_n(\tau) = \begin{cases} c_k \cos n\tau & \text{for} \quad \tau \in [\theta_{k,n}, \theta_{k+1,n}],\ k = 1, 2, ..., n-1, \\ 0 & \text{for all other } \tau \in [-\pi, \pi], \end{cases} \tag{54}$$

where $\theta_{k,n}$ is defined by (46) and $n \in N$.

It follows from (51) and (54) that $g_n \in C$ and $\|g_n\|_{\Lambda} < K$, $n \in N$. Meanwhile, combining (39), (40), (44), (45), (49), and (50), we have

$$|S_n(g)| > \left| \left( \int_{-\pi}^{-\pi/2n} + \int_{\pi/2n}^{\pi} \right) g(\tau) \frac{\cos n\tau}{\tau}\, d\tau \right| - K(r)\, \|g\|_{C[-\pi,\pi]} \tag{55}$$

for $n \in N$. Hence, according to (54) we have

$$|S_n(g_n)| > \sum_{k=1}^{n-1} \int_{\theta_{k,n}}^{\theta_{k+1,n}} \frac{\cos^2 n\tau}{\tau} \, d\tau - Kc_1 > K_1 \sum_{k=1}^{n-1} \frac{c_k}{k} - K_2. \tag{56}$$

Then by (52) and (56) $\|S_n\| \geqslant |S_n(g_n)|/\|g_n\|_\Lambda \to \infty$ as $n \to \infty$, and the necessity of statement (a) of Theorem 1 is proved. ▮

*Proof of Theorem* 2(a): *Sufficiency.* If a modulus of variation $v(n)$ satisfies (10), then $V[v] \subset HBV$ [1, Thm. 2, p. 232] (just set $\lambda_k = k$, $k \in N$). Hence, sufficiency of condition (10) immediately follows from Theorem 1.

*Necessity.* Suppose that condition (10) does not hold, i.e.,

$$\sum_{k=1}^{\infty} \frac{v(k)}{k^2} = \infty. \tag{57}$$

Applying Abel's transformation it is trivial to check that (57) implies

$$\sum_{k=1}^{\infty} \frac{v(k) - v(k-1)}{k} = \infty. \tag{58}$$

Again we apply the idea of unboundedness of the sequence of linear functionals $(S_n)_{n=1}^{\infty}$ defined on the Banach space $C \cap V[v]$ with norm (36). Following the construction of the counterexample for Theorem 1, let us consider the sequence of functions (54), where $c_k = v(k) - v(k-1)$, $k \in N$.

Since $g_n \in C$ and $\|g_n\|_V < K$ for $n \in N$, the rest of the proof follows from (56) and (58). ▮

*Proof of Theorem* 3(a): *Sufficiency.* It is known (see [6, Proof of Coro. 3, p. 479] and [12, p. 620]) that conditions (11), (12), and (13) are equivalent. At the same time, condition (11) implies the following inclusion [18, Thm. 1, p. 112]: $V_\Phi \subset HBV$. The rest of the proof follows from Theorem 1.

*Necessity.* Let us assume that condition (11) does not hold, i.e.,

$$\sum_{k=1}^{\infty} \Psi(1/k) = \infty. \tag{59}$$

As is obvious, we consider the same sequence of linear functionals $S_n$, but now on the Banach space $C \cap V_\Phi$ with norm (34). Again, we consider the sequence of functions defined by (54), where now $c_k = \psi(1/k)$, $k \in N$ (see Definition 4).

Since the functions $g_n$ are continuous by construction, let us estimate $\|g_n\|_\Phi$. Obviously, by the convexity of $\Phi$, we have

$$\begin{aligned} v_\Phi(g_n) &= \sum_{k=1}^{n-2} \Phi(\psi(1/k) - \psi(1/(k+1))) \\ &< \sum_{k=1}^{n-2} (\Phi(\psi(1/k)) - \Phi(\psi(1/(k+1)))) < \Phi(\psi(1)), \end{aligned}$$

and consequently $\|g_n\|_\Phi < K$ for $n \in N$. To complete the proof it suffices to estimate $S_n(g_n)$. From (56) we have

$$S_n(g_n) > K_1 \sum_{k=1}^{n-1} \frac{\psi(1/k)}{k} - K_2. \tag{60}$$

But it is known [12, conditions (1) and (5), pp. 619, 620] that condition (11) is also equivalent to the condition

$$\sum_{k=1}^{\infty} \frac{\psi(1/k)}{k} < \infty. \tag{61}$$

Hence, (59) implies the divergence of series (61), and by (60), the unboundedness of $S_n(g_n)$. The rest of the proof follows from the Banach–Steinhaus Theorem. ▌

As to assertions (b) of Theorems 1–3, it was mentioned in [9, p. 448] that for any odd function $g \in L$, $(S_n(g; 0))^{(2r)} = 0$ $(r, n \in Z_+)$, independent of the existence of a jump of a function $g$ at $\theta = 0$.

Theorems 4–6 are proved virtually identically, and so we omit the proofs.

*Proof of Theorem* 7. The following is an outline of the proof: First we establish the uniform equiconvergence of Fourier–Tchebycheff series and Fourier series with respect to the system of generalized Jacobi polynomials for an arbitrary function $f \in HBV$ strictly inside of the interval $[x_v, x_{v+1}]$, $v = 0, 1, ..., M$; then, applying Bernstein's inequality for polynomials, we obtain:

$$\|(S_n(\mathbf{w}; f; \cdot) - S_n^{(-1/2, -1/2)}(f; \cdot))^{(r)}\|_{C[\Delta(v; \varepsilon)]} = o(n^r) \tag{62}$$

for every $f \in HBV$, fixed $v = 0, 1, ..., M$, and $r \in Z_+$, where $\varepsilon \in (0, (x_{v+1} - x_v)/2)$, and $\Delta(v; \varepsilon)$ is defined by (19).

Finally, we shall prove identity (15) for the Fourier–Tchebycheff series of a function $f \in HBV$.

Taking into account (26), for the first step it suffices to show the following: Let $\mathbf{w} \in \mathrm{GJ}$, $\mathbf{w}(t)/h(t) = \rho(t)$, and suppose $f \in HBV$. Then

$$\|S_n(\mathbf{w}; f; \cdot) - S_n(\rho; f; \cdot)\|_{C[\Delta(v; \varepsilon)]} = o(1) \tag{63}$$

for every fixed $\varepsilon \in (0, (x_{v+1} - x_v)/2)$, $v = 0, 1, ..., M$.

Indeed, since $S_n(\mathbf{w}; Q_n; x) = Q_n(x)$ and the difference $R_n(\mathbf{w}; f; x) = f(x) - S_n(\mathbf{w}; f; x)$ is orthogonal with respect to the weight $\rho(t) = (1-t)^\alpha (1+t)^\beta \, |t - x_1|_1^\delta \cdots |t - x_M|^{\delta_M}$ to all $Q_n \in H_n$, we have

$$S_n(\mathbf{w}; f; x) - S_n(\rho; f; x)$$

$$= \int_{-1}^{1} R_n(\rho; f; t)[h(t) - h(x)] \, K_n(\mathbf{w}; x; t) \, \rho(t) \, dt = J. \tag{64}$$

Next, by the Christoffel–Darboux formula (20) and by virtue of (64) we get

$$\begin{aligned} J < \int_{-1}^{1} |R_n(\rho; f; t)| \, \frac{|h(t) - h(x)|}{|t - x|} \, &(|P_n(\mathbf{w}; x)| \, |P_{n-1}(\mathbf{w}; t)| \\ &+ |P_{n-1}(\mathbf{w}; x)| \, |P_n(\mathbf{w}; t)|) \, \rho(t) \, dt \\ = J^1 + J^2. \end{aligned} \tag{65}$$

Let $\varepsilon \in (0, (x_{v+1} - x_v)/2)$ be fixed and

$$x \in \Delta(v; \varepsilon). \tag{66}$$

Now, by (1), (21), (65), and (66), we have

$$\begin{aligned} J^1 &< K(\varepsilon) \left( \int_{[-1, 1] \setminus \Delta(v; \varepsilon/2)} + \int_{\Delta(v; \varepsilon/2)} \right) |R_n(\rho; f; t)| \, \frac{\omega(h; |t - x|)}{|t - x|} \, \bar{\rho}(t) \, dt \\ &= J^{11} + J^{12}, \end{aligned} \tag{67}$$

where the weight $\bar{\rho}(t) = (1-t)^{\bar{\alpha}} (1+t)^{\bar{\beta}} \, |t - x_1|^{\bar{\delta}_1} \cdots |t - x_M|^{\bar{\delta}_M}$ is defined as follows:

$$\bar{\alpha} = \begin{cases} \alpha, & \text{if} \quad -1 < \alpha < -1/2, \\ \alpha/2 - 1/4, & \text{if} \quad \alpha \geqslant -1/2, \end{cases} \tag{68}$$

$$\bar{\beta} = \begin{cases} \beta, & \text{if} \quad -1 < \beta < -1/2, \\ \beta/2 - 1/4, & \text{if} \quad \beta \geqslant -1/2, \end{cases} \tag{69}$$

$$\bar{\delta}_v = \begin{cases} \delta_v & \text{if} \quad -1 < \delta_v < 0, \\ \delta_v/2, & \text{if} \quad \delta_v \geqslant 0. \end{cases} \tag{70}$$

By (66), $|t-x|>\varepsilon/2$ for $J^{11}$. Consequently, using Hölder's inequality, we get

$$J^{11}<K(\varepsilon)\int_{[-1,1]\setminus\Delta(\nu;\varepsilon/2)}|R_n(\rho;f;t)|\,\bar{\rho}(t)\,dt$$

$$<K(\varepsilon)\int_{-1}^{1}|R_n(\rho;f;t)|\,(\rho(t))^{1/2}\,\{\bar{\rho}(t)(\rho(t))^{-1/2}\}\,dt$$

$$<K(\varepsilon)\left(\int_{-1}^{1}(R_n(\rho;f;t))^2\,\rho(t)\,dt\right)^{1/2}\left(\int_{-1}^{1}(\bar{\rho}(t))^2\,(\rho(t))^{-1}\,dt\right)^{1/2}. \tag{71}$$

But $(\bar{\rho}(t))^2\,(\rho(t))^{-1}\in L$ (see (68)–(70)), and by the completeness of the system $\sigma(\rho)$ [17, Thm. 3.1.5, p. 40], we obtain

$$J^{11}=o(1). \tag{72}$$

Regarding $J^{12}$, by (21), (66), and (67) we have

$$J^{12}<K(\varepsilon)\int_{\Delta(\nu;\varepsilon/2)}|R_n(\rho;f;t)|\,\frac{\omega(h;|t-x|)}{|t-x|}\,dt. \tag{73}$$

Since for $f\in HBV$ $R_n(\rho;f;x)=o(1)$ for $x\in(-1,1)$, $x\neq x_1,...,x_M$, (this, for example, follows from (26) and Theorem 2 [18, p. 112]), then in view of (3), (23), and the Lebesgue Convergence Theorem [14, Thm. 15, p. 76], we obtain

$$J^{12}=o(1) \tag{74}$$

uniformly with respect to (66). Hence, combining (67), (72), and (74), we get $J^1=o(1)$. By an obvious similarity, $J^2=o(1)$ as well, and this in conjunction with (64) and (65) proves (63).

To prove (62) we apply Bernstein's inequality (cf. [17, Thm. 1.22.3, p. 5]): if $Q_n\in H_n$, then

$$|Q_n'(x)|\leqslant\frac{n}{\sqrt{1-x^2}}\,\|Q_n\|_{C[-1,1]} \tag{75}$$

for $x\in[-1,1]$ and $n\in N$.

Indeed, for a fixed $\varepsilon\in(0,(x_{\nu+1}-x_\nu)/2)$, $\nu=0,1,...,M$, by virtue of (26) and (63) we have

$$\varepsilon_n=\|S_n(\mathbf{w};f;\,\cdot\,)-S_n^{(-1/2,\,-1,\,2)}(f;\,\cdot\,)\|_{C[\Delta(\nu;\varepsilon/2)]}=o(1) \tag{76}$$

for an arbitrary $f\in HBV$.

Hence, applying (75) to the polynomial

$$Q_n(x) = S_n(\mathbf{w}; f; x) - S_n^{(-1/2,\,-1/2)}(f; x)$$

by (76) for $r = 1$ we obtain

$$\frac{1}{n}\,\|(S_n(\mathbf{w}; f;\,\cdot\,) - S_n^{(-1/2,\,-1/2)}(f;\,\cdot\,))'\|_{C[\Delta(\nu;\,\varepsilon)]} \leqslant K(\varepsilon)\,\varepsilon_n = o(1). \qquad (77)$$

The proof of (62) for $r > 1$ is an obvious repetition of the above procedure.

Thus, to complete the sufficiency part of the proof it is enough to show that identity (15) for the Fourier–Tchebycheff series is correct.

Indeed, for a given $f \in HBV$, let the function $g$ be defined by (30). Differentiating with respect to $x$ the obvious identity $S_n^{(-1/2,\,-1/2)}(f; x) = S_n(g; \theta)$, where $x = \cos\theta$, we obtain by induction the following representation ($r \in Z_+$):

$$(S_n^{(-1/2,\,-1/2)}(f; x))^{(2r+1)}$$

$$= -(1 - x^2)^{-r-1/2}\,(S_n(g; \theta))^{(2r+1)} + \sum_{i=1}^{2r} d_i(x)(S_n(g; \theta))^{(i)} \qquad (78)$$

for $\theta \in [0, \pi]$, where $d_i$, $i = 1, 2, ..., 2r$, are infinitely differentiable functions on $(-1, 1)$.

In addition,

$$\|(S_n(g;\,\cdot\,))^{(i)}\|_{C[-\pi,\,\pi]} = o(n^{2r+1}) \qquad (79)$$

for $i = 1, 2, ..., 2r$, $r \in N$, since $g \in W \subset L$. Hence taking into account that $f(x\pm) = g(\theta\mp)$, $\theta \in (0, \pi)$, (15) immediately follows from (8) and (62). Thus the sufficiency of Theorem 7 is proved.

As to the assertion of definitiveness of Theorem 7, let us suppose that a sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$ is such that condition (9) does not hold. Then we shall show the existence of a continuous function $\bar{f} \in \Lambda BV$ such that

$$\limsup_{n\to\infty} \frac{|(S_n(\mathbf{w}; \bar{f}; \bar{x}))^{(2r+1)}|}{n^{2r+1}} > 0 \qquad (80)$$

for some fixed $\bar{x} \in (-1, 1)$, $x \neq x_1, ..., x_M$, and thus identity (15) will not hold.

It is known [2, Coro. 4.2, p. 51] that if a weight $\mathbf{w} \in \mathrm{GJ}$ also satisfies conditions (16), then

$$\|S_n(\mathbf{w}; f;\,\cdot\,) - S_n^{(-1/2,\,-1/2)}(f;\,\cdot\,)\|_{C[\Delta(\nu;\,\varepsilon)]} = o(1) \qquad (81)$$

for every $f \in C$ and fixed $\varepsilon \in (0, (x_{\nu+1} - x_\nu)/2)$, $\nu = 0, 1, ..., M$. Hence the combination of (81) and Bernstein's inequality (75) guarantees (62) *for every* $f \in C$, which in conjuncture with (78) and (79) implies

$$\frac{1}{n^{2r+1}} |(1-x^2)^{r+1/2} (S_n(\mathbf{w}; f; x))^{(2r+1)} + (S_n(g; \theta))^{(2r+1)}| = o(1)$$

for any fixed $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, where the function $g$ is related to the function $f$ via the formula (30). Then it suffices to consider the expression above for $\bar{x} = \cos \bar{\theta} \neq x_0, x_1, ..., x_{M+1}$ and the function $g_0(\theta - \bar{\theta})$, where $g_0$ is the function constructed for the counterexample of Theorem 1. ∎

Theorems 8 and 9 are proved similarly, and so we omit the details.

*Remark.* It is easy to check that a statement similar to the conclusion (b) of Theorem 1 is correct for Fourier–Tchebycheff series. Indeed, for any odd function $(1-t^2)^{-1/2} f \in L$, $(S_n^{(-1/2, -1/2)}(f; 0))^{(2r)} = 0$ for $n \geqslant 2r$, $r \in N$, independent of the existence of a jump of the function $f$ at $x = 0$. This follows from the fact that a Tchebycheff polynomial $P_n^{(-1/2, -1/2)}(x)$ is an even or odd function depending on whether its degree $n \in Z_+$ is even or odd [17, formula (4.1.3), p. 59].

*Proof of Corollary* 1. Since

$$|(S_n(\mathbf{w}; f; x))^{(r)}| \leqslant \sum_{k=r}^{n} |a_k(\mathbf{w}; f)| \, |(P_k(\mathbf{w}; x))^{(r)}|$$

for $r \in Z_+$, by virtue of Theorem 7 it is sufficient to show that

$$|(P_k(\mathbf{w}; x))^{(r)}| \leqslant K(x) \, k^r \tag{82}$$

for a fixed $x \in (-1, 1)$, $x \neq x_1, ..., x_M$, and an arbitrary $k \in N$. But (82) is an easy consequence of (21) and Bernstein's inequality (75). In the case when a weight $\mathbf{w} \in \mathrm{GJ}$ also satisfies condition (18), then inequality (82) holds for all $x \in (-1, 1)$ and the last implies that $f \in C$. ∎

Corollaries 2 and 3 are proved analogously.

## ACKNOWLEDGMENT

The author thanks Professor T. Hagstrom for some very helpful suggestions.

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